SIMPLE-WAVE-TYPE SOLUTIONS OF THE EQUATIONS OF TWO-DIMENSIONAL GASDYNAMICS

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In this paper we classify the partially invariant solutions of rank $\beta = 1$ and invariance defect $\delta = 1$ of the system of equations of two-dimensional gasdynamics.

1. A group classification of the equations of gasdynamics according to the function $p = f(\rho, S)$, specifying the equation of state, was carried out in [1]. In this paper we consider the system of differential equations of two-dimensional gasdynamics

$$
\begin{gathered}
d\mathbf{u} / dt + \rho^{-1}\nabla p = 0 \quad (d / dt = \partial / \partial t + u\cdot\nabla) \\
d\rho / dt + \rho \operatorname{div} \mathbf{u} = 0, \; dp / dt + \rho c^2 \operatorname{div} \mathbf{u} = 0 \quad (c^2 = \partial f / \partial \rho)
\end{gathered} \tag{1.1}
$$

Here, p is the pressure, ρ is the density, S is the entropy, and $\mathbf{u} = (u, v)$ is the velocity vector; they are all required functions of the independent variables x, y, and t. With $f(\rho, S)$ arbitrary, the widest Lie group of point transformations for system (1.1) is of order seven, and the basis of the corresponding Lie algebra L_7 consists of the operators [1]

$$
\begin{gathered}
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y} \\
X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v} \\
X_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad X_7 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}
\end{gathered} \tag{1.2}
$$

A knowledge of the group admitted by a system of differential equations enables its invariant and partially invariant solutions to be isolated. Our aim in this paper is to classify all the partially invariant solutions of rank $\beta = 1$ and invariance defect $\delta = 1$. Here, $\beta = 1$ implies that there are four functions (invariants of the three-parameter subgroup) such that three of them are expressible in terms of the fourth. A defect $\delta = 1$ implies, in general, that the obtained solutions contain one arbitrary function. Simple waves belong to this type of solution, so that we in fact describe as simple-wave type solutions the partially invariant solutions of rank $\beta = 1$ and invariance defect $\delta = 1$. In accordance with [2], such solutions must be constructed in third-order subgroups. A system of classes of dissimilar third-order subgroups was found in [2], which may be reduced for computational convenience to twenty different types of class. The relevant table is given below.

TABLE 1

	Operators				Operators		
1	X_1	X_2	X_3	11	X_2	X_4	$X_6 + \alpha X_5$
2	X_1	X_2	X_4	12	X_2	X_4	$X_1 + X_5$
3	X_1	X_2	$X_3 + X_4$	13	X_2	X_4	X_5
4	X_1	X_2	$X_6 + \alpha X_4$	14	X_2	$X_3 + X_4$	$X_1 + X_5$
5	X_1	X_6	X_7	15	X_4	X_5	$X_7 + \alpha X_6$
6	X_2	X_3	$X_7 + \alpha X_6$	16	X_4	X_5	$X_2 + X_7 + \alpha X_6$
7	X_2	X_3	$X_1 + X_7$	17	X_4	X_5	X_6
8	X_2	X_3	$X_1 + X_5$	18	X_4	$X_3 + \alpha X_2$	$X_6 + \alpha X_5$
9	X_2	X_3	X_4	19	$X_2 + X_5$	X_7	$X_3 - X_4$
10	X_2	X_3	$X_6 + \alpha X_4$	20	$X_2 + X_5$	X_3	$X_4 + \alpha X_2$

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The algorithm described in [1] was used for obtaining the partially invariant solutions. By means of this algorithm, the initial system (1.1) is split into a system (1.1)/H, in which only invariants appear, and a passive system P, which includes invariants and parametric functions. Of the solutions obtained, those are eliminated which can be obtained as invariant solutions of the same rank $\beta = 1$; the problem of finding these latter is much simpler. In the case in which partially invariant solutions can be obtained as invariant solu-tions can be obtained as invariant solutions, reduction of the partially invariant to invariant solutions is tions, reduction of the partially invariant to invariant solutions is said to take place. For first-order systems, sufficient conditions for reduction are given by the following theorem, from Ovsyannikov.

Theorem. If expressions for all the first-order derivatives of the parametric functions can be obtained from the passive system, then there exists, for every partially invariant H-solution, a subgroup $H' \subset H$ such that this solution is an invariant H'-solution of the same rank.

2. The only solutions of interest are those in which the pressure is not identically constant, since the general solution of system (1.1) can be found when $p \equiv \text{const}$. The following results may be obtained for the simple-wave type solutions:

2.1. As in the case of simple waves, only isentropic partially invariant solutions exist, not reducible to invariant solutions. A similar property was proved by Ovsyannikov for double waves [3];

2.2. Let I^{τ} ($\tau = 1, \ldots, 4$) denote the complete set of invariants of a three-parameter subgroup and let h be the rank of the Jacobian of the functions I^{τ} with respect to the variables u, v, p, and ρ. Since the pressure p and density ρ are invariants of the group admitted by system (1.1) with $f(\rho, S)$ arbitrary, it is obvious that h can only take the values 3 and 4. The following result holds for all partially invariant solutions of rank $\beta = 1$ and invariance defect $\delta = 1$, apart from simple waves: if $h = 4$, then irreducible partially invariant solutions can only exist for special equations of state; and on the contrary, if $h = 3$, irreducible solutions of the simple-wave type exist for any function $f(\rho)$.

3. Our results are summarized below. The following notation is employed: a, b, c_0, α, u_0, v_0, p_0, ρ_0 are arbitrary constants, and Φ is an arbitrary function. The numbers of the solutions are the same as the subgroup numbers in the table.

3.1. Solution 1 consists of simple waves. It has been investigated in detail and need not be considered here.

3.2. There are no irreducible solutions of the simple wave type in subgroups 5, 6, 10, 13, 16 ($\alpha \neq 0$), 19, and 20.

Notice that the following three subclasses can be isolated in the class of irreducible simple wave type solutions:

a) solutions in which the equation of state $p = f(\rho)$ is arbitrary, u(t, x, y) is linear in the variable x, and v and ρ are in general functions of the variables t and y;

b) solutions in which the equation of state is arbitrary, but **u** (t, x, y) is a nonlinear vector function in all the independent variables;

c) solutions with a special equation of state; solutions of subclass a) exist in subgroups 2, 9, 11, and 12, and are as follows.

3.3. Solution 2. Let

$$\varphi(\rho) = \left(2\int_{\rho}^{a} \frac{dp}{\rho}\right)^{1/2}, \qquad \psi(\rho) = \int_{b}^{\rho} \left(\frac{\varphi'}{\varphi\rho} + \frac{1}{\rho^2}\right) d\rho, \qquad \omega(\rho) = -\frac{1}{\rho\varphi}$$

Then this solution is such that the $\rho = \rho(y)$ is implicitly defined by

$$y = c_0 \psi(\rho)$$

while the functions u(t, x, y) and v(y) are given in terms of the known function $\rho(y)$ by

$$v = \varphi(\rho), \qquad u = \rho\varphi\left[\Phi(t - c_0\omega(\rho)) - \frac{x}{c_0}\right]$$

3.4. Solution 9:

$$\rho = \frac{b}{t+a}, \quad v = 0, \quad u = \frac{x + \Phi(y)}{t+a}$$

3.5. Solution 11:

$$v = \alpha \ln t + \omega(z), \quad \rho = \rho(z), \quad z = y/t - \alpha \ln t$$

The functions $\rho(z)$ and $\omega(z)$ are found from the system of ordinary equations

$$\omega'(\omega - z - \alpha) + \frac{\rho'}{\rho}\frac{df}{d\rho} + \alpha = 0$$

$$(\omega - z - \alpha)\left(\omega'' - 2\frac{\rho'}{\rho} - \omega'\frac{\rho'}{\rho}\right) + (\omega - z - \alpha)^2\left[\left(\frac{\rho'}{\rho}\right)' - \left(\frac{\rho'}{\rho}\right)^2\right] = \omega' + \omega'^2$$

The function u(t, x, y) is restored by a quadrature with respect to the known functions $\rho(z)$ and $\omega(z)$:

$$u = -\frac{x}{t}\left[\omega' + \frac{\rho'}{\rho}(\omega - z - \alpha)\right] + \rho \exp\left(\int \frac{\omega' dz}{\omega - z - \alpha}\right) \Phi\left(\ln t - \int \frac{dz}{\omega - z - \alpha}\right)$$

3.6. Solution 12:

$$\rho = \rho(z), \quad v = t + \omega(z)$$

$$z = 2y - t^2, \quad \omega = (a - \sigma(\rho) - z)^{1/2}, \quad \sigma(\rho) = 2\int \frac{dp}{\rho}$$

is a known function, while $\rho(z)$ satisfies the equation

$$2(a - \sigma - z)^2\left[\left(\frac{\rho'}{\rho}\right)' - \left(\frac{\rho'}{\rho}\right)^2\right] + (a - \sigma - z)\left[\frac{\rho'}{\rho}(\sigma'\rho' + 1) - (\sigma''\rho'^2 + \sigma'\rho'')\right] = (\sigma'\rho' + 1)^2$$

$$\sigma' \equiv d\sigma/d\rho, \qquad \sigma'' \equiv d^2\sigma/d\rho^2$$

The function u(t, x, y) is restored by a quadrature via $\rho(z)$ and $\omega(z)$:

$$u = -2x\left(\omega' + \frac{\rho'}{\rho}\omega\right) + \rho\omega\Phi\left(t - \frac{1}{2}\int \frac{dz}{\omega}\right)$$

The subclass b) consists of a single solution, obtained in subgroup 15, in which, in view of the irreducibility condition, we have to put $\alpha = 0$. This solution will be quoted below.

3.7. Solution 15:

$$\rho = \frac{1}{at + bt^2}, \quad u = \frac{x}{t} + \frac{a}{t}\cos\theta, \quad v = \frac{y}{t} + \frac{a}{t}\sin\theta$$

The function θ is given implicitly by

$$\Phi(\lambda, \theta - \gamma) = 0$$

$$\lambda^2 = \frac{r^2 + a^2}{t^2} + 2\frac{r}{t}\cos(\theta - \varphi)\left(\frac{a}{t} + b\right) + 2b\frac{a}{t} + b^2$$

$$\gamma = \operatorname{arc\,tg}\frac{r\sin(\theta - \varphi)}{bt + a + r\cos(\theta - \varphi)}, \quad r^2 = x^2 + y^2, \quad \varphi = \operatorname{arc\,tg}\frac{y}{x}$$

Solutions of subclass c) exist in subgroups 3, 4, 7, 8, 14, 17, and 18, and are as follows.

3.8. Solution 3:

$$p = p_0 + \frac{a^2}{\rho_0} - \frac{a^2}{\rho}, \quad u = u_0 + y, \quad v = b - \frac{a}{\rho}$$

The function ρ (t, x, y) is obtained implicitly from

$$\frac{a}{\rho} + b\ln\left(\frac{a - b\rho}{\rho}\right) = \varphi(t, y) - x$$

a) $$\varphi(t, y) = \frac{y^2}{2b} + u_0\frac{y}{b} + \Phi(y - bt) \qquad (b \neq 0)$$

b) $$\varphi(t, y) = t(y + u_0) + \Phi(y) \qquad (b = 0)$$

3.9. Solution 4:

$$p = p_0 + \frac{a^2}{\rho_0} - \frac{a^2}{\rho}, \quad u = \alpha \ln y + u_0, \quad v = b - \frac{a}{\rho}$$

The function ρ (t, x, y) is given implicitly by

$$\frac{a}{\rho} + b \ln\left(\frac{a - b\rho}{\rho}\right) = \varphi(t, y) - \frac{\alpha x}{y}$$

a) $$\varphi(t, y) = \frac{\alpha^2}{2b} \ln^2 y + \frac{\alpha u_0}{b} \ln y + \Phi(y - bt) \qquad (b \neq 0)$$

b) $$\varphi(t, y) = \frac{\alpha t}{y}(\alpha \ln y + u_0) + \Phi(y) \qquad (b = 0)$$

3.10. Solution 7:

$$p = \frac{b^2 c_0}{2} \exp\left(-2\frac{c_0}{\rho}\right), \quad q = b \exp\left(-\frac{c_0}{\rho}\right), \quad \theta = t + a$$

The function ρ (r, φ, t) is given implicitly by

$$\Phi(l^2, \gamma - t) = 0$$

$$l^2 = \xi^2 + \eta^2, \quad \gamma = \operatorname{arc\,tg} \frac{\eta}{\xi}, \quad \xi = r \cos(t + a - \varphi)$$

$$\eta = r \sin(t + a - \varphi) - b \exp\left(-\frac{c_0}{\rho}\right)\left(\frac{c_0}{\rho} + 1\right)$$

$$r^2 = x^2 + y^2, \quad \varphi = \operatorname{arc\,tg} \frac{y}{x}, \quad q^2 = u^2 + v^2, \quad \theta = \operatorname{arc\,tg} \frac{v}{u}$$

3.11. Solution 8:

$$p = p_0 + \frac{a^2}{\rho_0} - \frac{a^2}{\rho}, \quad u = b - \frac{a}{\rho}, \quad v = t$$

$$\rho = a\left\{2\left[y - \frac{t^2}{2} - \Phi(x - bt)\right]\right\}^{-1/2}$$

3.12. Solution 14a:

$$p = c_0 - b^2\left[\frac{1}{a(b + a\rho)} + \frac{1}{\rho}\right]$$

$$u = y - \frac{t^2}{2} + \ln \frac{u_0 \rho}{a\rho + b}, \quad v = t + a + \frac{b}{\rho}$$

The function ρ (t, x, y) is given implicitly by

$$\Phi\left(z_2 - \frac{t^2}{2} - at,\ z_1 - z_2 t + \frac{t^3}{2} + \frac{at^2}{2} - (\ln u_0 + 1)t\right) = 0$$

$$z_1 = x + a \ln\left(\frac{b + a\rho}{\rho}\right) - \frac{b}{\rho}, \quad z_2 = y - \ln\left(\frac{b + a\rho}{\rho}\right) - \frac{a\rho}{b + a\rho}$$

Solution 14b:

$$p = c_0 - \frac{1}{a(1 + a\rho)}, \quad u = y - \frac{t^2}{2} + \ln \frac{b\rho}{1 + a\rho}, \quad v = t$$

The function ρ (t, x, y) is obtained from the relationship

$$\ln \frac{\rho}{1 + a\rho} + \frac{1}{1 + a\rho} - 1 = \varphi(x, t) - y$$

where φ(x, t) is also given in implicit form:

$$\Phi(\varphi - {}^1/_2 t^2,\ x - t\varphi + {}^1/_2 t^3 - (\ln b + 1)t) = 0$$

3.13. Solution 17a:

$$u = x / t + a\sqrt{\rho}, \quad v = y / t, \quad p = {}^1/_8 a^2\rho^2 + b$$

The function $\rho(t, x)$ is given implicitly:

$$\Phi\,(t\sqrt{\rho},\ x / t + {}^3/_2 a\sqrt{\rho}) = 0$$

Solution 17b:

$$u = x / t, \qquad v = y / t + a\sqrt{\rho}, \qquad p = {}^1/_8 a^2\rho^2 + b$$

The function ρ (y, t) is given implicitly by

$$\Phi\,(t\sqrt{\rho},\ y / t + {}^3/_2 a\sqrt{\rho}) = 0$$

3.14. Solution 18. An irreducible solution only exists when $\alpha = 0$:

$$p = {}^1/_3 a^2\rho^3 + b, \qquad u = x / t + a\rho, \qquad v = v_0$$

The function ρ (x, t) is given implicitly by

$$\Phi\,(t\rho,\ x / t + 2a\rho) = 0$$

4. Examination of the simple-wave type solutions showed that solutions 9 and 15 can be generalized if the pressure and density in system (1.1) are assumed to depend on time only. On putting p = p(t) and $\rho = \rho$ (t) in system (1.1) and discounting the nonisentropic case (in which the pressure and density are constant), the following system of equations is obtained:

$$u_t + uu_x + vu_y = 0, \quad v_t + uv_x + vv_y = 0, \quad \mu\,(t) + u_x + v_y = 0 \tag{4.1}$$

Here the notation μ (t) $= \rho'/\rho$ has been used for typographical simplicity. Notice that, on taking p = p(t) and $\rho = \rho$(t), the partially invariant solution of rank $\beta = 1$ and invariance defect $\delta = 2$ will be discovered from the subgroup $H = \langle X_2, X_3, X_4, X_5\rangle$. Since the invariance defect $\delta = 2$ here, this solution should differ from those of the simple-wave type in containing two arbitrary functions.

Let us examine system (4.1). The first two equations of the system can be integrated in the form

$$x - tu = \varphi\,(u, v), \qquad y - tv = \psi\,(u, v) \tag{4.2}$$

Here φ and ψ are arbitrary functions. To solve the third equation, the independent variables are changed: (x, y, t) → (u, v, t). It is easily shown that, when $\mu(t) = -(t + a)^{-1}$, the functions u and v are functionally dependent. To prove this, we differentiate the first and second equations of system (4.1) with respect to x and y, respectively, add the results, and then use the third equation of the system. The case $\mu(t) = -(t + a)^{-1}$ or $\rho = (a + bt)^{-1}$ ($\rho'/\rho = \mu$) must therefore be regarded as singular, in the sense of the mapping (x, y, t) → (u, v, t). On transforming to the variables u, v, and t, and recalling (4.2), the last equation of (4.1) can be written as

$$\mu\,(t) = -\frac{2t + \varphi_u + \psi_v}{\varphi_u\psi_v - \varphi_v\psi_u + t\,(\varphi_u + \psi_v) + t^2} \tag{4.3}$$

It is easily shown that, by virtue of (4.3), μ(t) must satisfy the equation

$$\mu'' - 3\mu\mu' + \mu^3 = 0 \qquad (\lambda\,(t) = \exp\,[-\textstyle\int \mu dt])$$

This equation can be integrated (the substitution is shown in parentheses), and its solution is

$$\mu = -\frac{b + 2c_0 t}{a + bt + c_0 t^2} \tag{4.4}$$

Since μ(t) $= \rho'/\rho$, we get the following expression for ρ(t):

$$\rho = \frac{1}{a + bt + c_0 t^2} \quad (c_0 \neq 0)$$

Substituting (4.4) for μ(t) into (4.3) and equating coefficients of like powers of t, we obtain the system of equations

$$\varphi_u + \psi_v = b / c_0, \qquad \varphi_u\psi_v - \varphi_v\psi_u = a / c_0 \tag{4.5}$$

In short, with the assumption that $p = p(t)$ and $\rho = \rho(t)$, we find that

$$\rho = \frac{1}{a + bt + c_0 t^2} \tag{4.6}$$

and when $c_0 \neq 0$ the solution of system (4.1) must be

$$x - tu = \varphi(u, v), \quad y - tv = \psi(u, v)$$

where the functions φ and ψ satisfy system (4.5).

Comparing expression (4.6) for $\rho(t)$ with the analogous expressions in solutions 9 and 15, it can be seen that the generalization of solution 9 will be obtained with $c_0 = 0$ in (4.6). Solution 15 can be generalized by putting $a = 0$ in (4.6). Notice that it is in precisely these cases that complete integration of system (4.1) is possible. If $\rho = (a + bt)^{-1}$ and, in view of the functional dependence of u and v, we put $v = F(u)$, integration of system (4.1) gives the solution

$$\begin{gathered} u = u(t, x, y), \qquad \rho = \frac{1}{a + bt} \\ F(u) = v = \frac{by}{a + bt} + \frac{b}{a + bt}\Phi(bx - u(bt + a)) \end{gathered} \tag{4.7}$$

Here, $\Phi(z)$ and $v = F(u)$ are arbitrary functions. If we put $a = 0$ (the case of solution 15), it follows from the second of Eq. (4.5) that φ and ψ are also functionally dependent. Putting $\psi = F(\varphi)$ and integrating system (4.5), the following solution is obtained:

$$\begin{gathered} x - tu = \varphi(u, v), \quad \rho = \frac{1}{bt + c_0 t^2} \\ F(\varphi) = y - tv = \frac{b}{c_0} v + \Phi(bu - c_0\varphi) \end{gathered} \tag{4.8}$$

The functions $\psi = F(\varphi)$ and $\Phi(z)$ are arbitrary. In solutions (4.7) and (4.8) we have an arbitrary isentropic equation of state. It was verified that no generalization is obtained for simple-wave type solutions of subclass a) [similar to the generalizations (4.7) and (4.8)] if we put

$$\begin{gathered} u = a(t, y)x + b(t, y), \quad v = v(t, y) \\ \rho = \rho(t, y), \quad p = f(\rho) \end{gathered}$$

in system (1.1).

In conclusion, it seems worth pointing out the importance of an investigation of the simple-wave type solutions in the context of concrete gasdynamic problems.

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